

COVERING SYSTEMS WITH RESTRICTED DIVISIBILITY

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ABSTRACT. We prove that every distinct covering system has a modulus divisible by either 2 or 3.

1. INTRODUCTION

A covering system of congruences is a collection

$$a_i \bmod m_i, \quad i = 1, 2, \dots, k$$

such that every integer satisfies at least one of them. A covering system is distinct if the moduli m_i are distinct and greater than 1. An old problem of Erdős and Selfridge asks whether there exists a distinct covering system of congruences with all moduli odd. Improving work of Simpson and Zeilberger [10], Guo and Sun [4] proved that an odd covering system with square-free moduli involves at least 22 primes. We modify the first author's solution of the minimum modulus problem for covering systems [5] to prove the following further quantitative result towards the odd covering problem.

Theorem 1. *Every distinct covering system of congruences has a modulus divisible by either 2 or 3.*

2. SET-UP

Suppose given a finite set of moduli \mathcal{M} , and, for each $m \in \mathcal{M}$, a set of residues \mathbf{a}_m modulo m . Let

$$Q = \text{LCM}(m : m \in \mathcal{M})$$

and

$$R = \mathbb{Z} \setminus \bigcup_{m \in \mathcal{M}} (\mathbf{a}_m \bmod m),$$

which is a set defined modulo Q . One way to show that the congruences

$$(\mathbf{a}_m \bmod m), \quad m \in \mathcal{M}$$

do not cover the integers is to give a positive lower bound for the density of R . The proof of Theorem 1 gives such a lower bound, although quantitatively it estimates some related quantities.

If we let $\mathbb{Z}/Q\mathbb{Z}$ have the uniform probability measure, then the density of R is equal to its probability. For $m \in \mathcal{M}$ let A_m be the event $(\mathbf{a}_m \bmod m)$, which has probability $\frac{|\mathbf{a}_m|}{m}$, and extend this to $m|Q$ with $m \notin \mathcal{M}$ by setting $A_m = \emptyset$ for these m . Then

$$(1) \quad \mathbf{P}(R) = \mathbf{P} \left(\bigcap_{m|Q} A_m^c \right).$$

A familiar argument (the Chinese Remainder Theorem) implies that A_m is independent of any set of congruences to moduli co-prime to m . Thus a valid dependency graph for the events $\{A_m : m|Q\}$ has edge (m, m') if and only if $\text{GCD}(m, m') > 1$.

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A family of results connected to the Lovász Local Lemma give worst-case lower bounds for the probability of an intersection as in (1), taking as input only the events' probabilities and their dependency graph. In principle we could hope to prove Theorem 1 by directly applying one of these results to claim that the uncovered set always has a non-zero density, but, as we will see, such a lower bound cannot be given, and further input is needed. Two methods of Lovász type do figure into our argument, however, as we will describe.

Given the problem of estimating from below the probability of the intersection of the complements of some events given only their probabilities and their dependency graph, the best possible estimate has been given by Shearer [9]. The estimate is best possible in the sense that the argument constructs a probability space and events having the prescribed probabilities and dependency graph, and such that the lower bound holds with equality. However, the condition with which Shearer's result holds can be difficult to verify, and so the following result is useful because it is easy to check. Note that this is essentially due to [10] in this context.

Theorem 2 (Shearer-type theorem). *Suppose we have a probability space. Let $[n] = \{1, 2, \dots, n\}$, and assume that for each $1 \leq i \leq n$ there is a weight π_i assigned, satisfying $1 \geq \pi_1 \geq \pi_2 \geq \dots \geq \pi_n \geq 0$. Let the sets $\emptyset \neq T \subset [n]$ index events A_T each having probability*

$$0 \leq \mathbf{P}(A_T) \leq \prod_{t \in T} \pi_t := \pi_T.$$

Assume that A_T is independent of $\sigma(\{A_S : S \subset [n], S \cap T = \emptyset\})$, so that a valid dependency graph for the events $\{A_T : \emptyset \neq T \subset [n]\}$ has an edge between $S \neq T$ whenever $S \cap T \neq \emptyset$.

Define $\rho(\emptyset) = 1$, and given $\emptyset \neq T \subset [n]$, set (put an arbitrary total ordering $<$ on $2^{[n]}$ to avoid confusion)

$$\rho(T) = 1 - \sum_{\emptyset \neq S_1 \subset T} \pi_{S_1} + \sum_{\substack{\emptyset \neq S_1, S_2 \subset T \\ S_1 < S_2 \text{ disjoint}}} \pi_{S_1} \pi_{S_2} - \sum_{\substack{\emptyset \neq S_1, S_2, S_3 \subset T \\ S_1 < S_2 < S_3 \text{ disjoint}}} \pi_{S_1} \pi_{S_2} \pi_{S_3} + \dots$$

Suppose that $\rho([1]) \geq \rho([2]) \geq \dots \geq \rho([n]) > 0$. Then for any $\emptyset \neq T \subset [n]$,

$$(2) \quad \mathbf{P} \left(\bigcap_{\emptyset \neq S \subset T} A_S^c \right) \geq \rho(T) > 0$$

and, for any $T_1 \subset T_2 \subset [n]$,

$$(3) \quad \frac{\mathbf{P} \left(\bigcap_{\emptyset \neq S \subset T_2} A_S^c \right)}{\mathbf{P} \left(\bigcap_{\emptyset \neq S \subset T_1} A_S^c \right)} \geq \frac{\rho(T_2)}{\rho(T_1)}.$$

We prove a slightly more general version of this theorem in Appendix C.

To apply the Shearer-type theorem in the context of Theorem 1, order the primes greater than 3 as $p_1 = 5, p_2 = 7, p_3 = 11, \dots$. Suppose we are given a distinct congruence system with moduli formed with the primes p_1, \dots, p_n . Identify $S \subset [n]$ with the square-free number $m_S = \prod_{i \in S} p_i$ and form the event A_S which is the union of all congruences having square-free part m_S ,

$$A_S = \bigcup_{m: \text{sqf}(m)=m_S} (a_m \bmod m),$$

where $\text{sqf}(m) = \prod_{p:p|m} p$. Then A_S is an event with probability

$$\mathbf{P}(A_S) < \prod_{i \in S} \frac{1}{p_i - 1}.$$

In particular, we may appeal to Theorem 2 with $\pi_i = \frac{1}{p_i-1}$. Arguing in this way, we may check that there is no covering composed of only the primes between 5 and 631, but at this point, the Shearer function becomes negative, and no further result can be drawn from that estimate.

What allows us to make further progress is that, within the range in which Shearer's theorem holds, estimate (3) of Theorem 2 gives substantial information about the structure of the uncovered set. To see this, suppose that we have a congruence system as above with uncovered set R , and that Theorem 2 applies. We can estimate the proportion of the set R that lies in a given congruence class $(b \bmod m)$ for $m|Q$ by

$$\begin{aligned} \frac{\mathbf{P}((b \bmod m) \cap R)}{\mathbf{P}(R)} &= \frac{\mathbf{P}\left((b \bmod m) \cap \bigcap_{m' \in \mathcal{M}, m'|Q} (a_{m'} \bmod m')^c\right)}{\mathbf{P}\left(\bigcap_{m' \in \mathcal{M}, m'|Q} (a_{m'} \bmod m')^c\right)} \\ &\leq \mathbf{P}((b \bmod m)) \frac{\mathbf{P}\left(\bigcap_{m' \in \mathcal{M}, m'|Q, (m, m')=1} (a_{m'} \bmod m')^c\right)}{\mathbf{P}\left(\bigcap_{m' \in \mathcal{M}, m'|Q} (a_{m'} \bmod m')^c\right)} \\ &= \frac{1}{m} \frac{\mathbf{P}\left(\bigcap_{m' \in \mathcal{M}, m'|Q, (m, m')=1} (a_{m'} \bmod m')^c\right)}{\mathbf{P}\left(\bigcap_{m' \in \mathcal{M}, m'|Q} (a_{m'} \bmod m')^c\right)}. \end{aligned}$$

The ratio of probabilities on the right is bounded by the relative conclusion (3) of Theorem 2, which gives a ratio of $\frac{\rho([n] \setminus S_m)}{\rho([n])}$ where $[n]$ again represents the full set of primes dividing Q , and S_m is those primes from $[n]$ which divide m . Thus

$$\frac{\mathbf{P}((b \bmod m) \cap R)}{\mathbf{P}(R)} \leq \frac{1}{m} \frac{\rho([n] \setminus S_m)}{\rho([n])}.$$

If S_m is such that $\rho([n] \setminus S_m) \approx \rho([n])$ then we deduce that R is almost uniformly distributed across residues modulo m .

We summarize the above discussion in the following Theorem.

Theorem 3. *Let $p_1 < p_2 < \dots < p_n$ be a sequence of primes, and let weights π_1, \dots, π_n given by $\pi_i = \frac{1}{p_i-1}$. For a subset $S \subset [n]$ identify S with $q_S = \prod_{p \in S} p$, and write $\rho(q) = \rho(q_S) = \rho(S)$ for the Shearer function associated to S with weights π_i , as in Theorem 2.*

Suppose that $\rho(p_1) \geq \rho(p_1 p_2) \geq \dots \geq \rho(p_1 p_2 \dots p_n) > 0$. Then any distinct congruence system with moduli composed only of p_1, \dots, p_n does not cover the integers. Moreover, if R is the uncovered set and if m is a modulus composed of primes corresponding to a set $S \subset [n]$ then

$$(4) \quad \max_{b \bmod m} \frac{|R \cap (b \bmod m)|}{|R|} \leq \frac{1}{m} \frac{\rho(q_{[n] \setminus S})}{\rho(q_{[n]})}.$$

Although the sieving problem described in Theorem 1 concerns systems of congruences in which each congruence set \mathbf{a}_m has size 0 or 1, in the course of our argument we consider congruences with sets \mathbf{a}_m of variable size. In this situation the condition of Theorem 2 becomes unwieldy and we appeal instead to the following Theorem, which follows from an improved form of the Lovász Local Lemma due to [1], see also [8].

Theorem 4. *Let $\mathcal{N} \subset \mathbb{N}_{>1}$ be a finite collection of moduli whose prime factors are drawn from a set of primes \mathcal{P} . Let $\text{LCM}(n : n \in \mathcal{N}) = Q$. Suppose that for each $n \in \mathcal{N}$ a*

collection of residues $\mathbf{a}_n \bmod n$ is given. Write

$$R = \mathbb{Z} \setminus \bigcup_{n \in \mathcal{N}} (\mathbf{a}_n \bmod n).$$

Suppose that there exist weights $\{x_p\}_{p \in \mathcal{P}}$ with $x_p \geq 0$, which satisfy the constraints

$$\forall p \in \mathcal{P}, \quad x_p \geq \sum_{n \in \mathcal{N}: p|n} \frac{|\mathbf{a}_n \bmod n| \prod_{p'|n} (1 + x_{p'})}{n}.$$

Then the density of R is at least

$$(5) \quad \frac{|R \bmod Q|}{Q} \geq \exp \left(- \sum_{n \in \mathcal{N}} \frac{|\mathbf{a}_n \bmod n| \prod_{p|n} (1 + x_p)}{n} \right) > 0.$$

Also, for any $n \in \mathcal{N}$,

$$(6) \quad \max_{b \bmod n} \frac{|R \cap (b \bmod n) \bmod Q|}{|R \bmod Q|} \leq \frac{\exp \left(\sum_{p|n} x_p \right)}{n}.$$

Remark. Conclusion (5) corresponds to (2) of Theorem 2, and (6) corresponds to (4).

If we write \underline{x} for $\{x_p\}_{p \in \mathcal{P}}$ and $\underline{G}(\underline{x})$ for

$$G_p(\underline{x}) = \sum_{n \in \mathcal{N}: p|n} \frac{|\mathbf{a}_n \bmod n| \prod_{p'|n} (1 + x_{p'})}{n}$$

then the condition of Theorem 4 equivalently asks for a non-negative ($\underline{x} \geq \underline{0}$) fixed point $\underline{G}(\underline{x}) = \underline{x}$, which is relatively easy to determine. Thus, although Theorem 4 is strictly weaker than Theorem 2, it is useful since it is more easily applied.

A proof and further discussion of Theorem 4 is given in Section 4.

3. OVERVIEW OF ARGUMENT

We now give an overview of our argument. As the structure is similar to that of the minimum modulus problem we refer the proofs of some background statements to [5].

We assume given a congruence system with finite set of moduli

$$\mathcal{M} \subset \{m > 1, (m, 6) = 1\},$$

together with a residue class $a_m \bmod m$ for each $m \in \mathcal{M}$. We let

$$Q = \text{LCM}(m : m \in \mathcal{M}),$$

and set

$$R = \mathbb{Z} \setminus \bigcup_{m \in \mathcal{M}} (a_m \bmod m)$$

for the set left uncovered by the congruence system. Theorem 1 follows by showing that the density of R is positive.

To estimate the density of R we appeal to Lovász Local Lemma-type arguments of the previous section. These arguments, however, only apply to estimate the density of sets left uncovered by congruence systems whose moduli are composed of a limited number of primes, and so we break the estimate for the density of R into stages.

Let $P_0 = 4 < P_1 < P_2 < \dots$ be a sequence of real numbers (not equal to prime integers). Let $Q_0 = 1$ and, for $i \geq 1$,

$$Q_i = \prod_{p^j \parallel Q, p < P_i} p^j$$

be the part of Q composed of primes less than P_i . We let $\mathcal{M}_i = \{m \in \mathcal{M} : m|Q_i\}$ be the P_i -smooth moduli in \mathcal{M} , and we let the set of ‘new factors’ be

$$\mathcal{N}_i = \{n > 1 : n|Q_i, p|n \Rightarrow P_{i-1} < p \leq P_i\}.$$

Notice that each $m \in \mathcal{M}_{i+1} \setminus \mathcal{M}_i$ has a unique factorization as $m = m_0 n$ with $m_0|Q_i$ and $n \in \mathcal{N}_{i+1}$.

We consider the sequence of sets $\mathbb{Z} = R_0 \supset R_1 \supset \dots$,

$$\forall i \geq 1, \quad R_i = \mathbb{Z} \setminus \bigcup_{m \in \mathcal{M}_i} (a_m \bmod m).$$

Since $R_i = R$ eventually, it will suffice to show that R_i is non-empty for each i .

The set R_i is defined modulo Q_i . Viewing $\mathbb{Z}/Q_{i+1}\mathbb{Z}$ as fibered over $\mathbb{Z}/Q_i\mathbb{Z}$ we note that

$$R_{i+1} = R_i \setminus \bigcup_{m \in \mathcal{M}_{i+1} \setminus \mathcal{M}_i} (a_m \bmod m),$$

so that we may view R_{i+1} as cut out from the fibers $(r \bmod Q_i)$, $r \in R_i$, by congruences to moduli in $\mathcal{M}_{i+1} \setminus \mathcal{M}_i$. Given $r \in R_i$ and $m \in \mathcal{M}_{i+1} \setminus \mathcal{M}_i$, factor $m = m_0 n$ with $m_0|Q_i$ and $n \in \mathcal{N}_{i+1}$. Then the congruence $(a_m \bmod m)$ meets $(r \bmod Q_i)$ if and only if $r \equiv a_{m_0 n} \bmod m_0$, and when it does so, it intersects in a single residue class modulo nQ_i . Thus, grouping together moduli according to common new factor $n \in \mathcal{N}_{i+1}$ we find

$$R_{i+1} \cap (r \bmod Q_i) = (r \bmod Q_i) \setminus \bigcup_{n \in \mathcal{N}_{i+1}} A_{n,r},$$

with

$$A_{n,r} = (r \bmod Q_i) \cap \bigcup_{\substack{m_0|Q_i \\ m_0 n \in \mathcal{M}_{i+1}}} (a_{m_0 n} \bmod m_0 n).$$

After translating and dilating $(r \bmod Q_i)$ to coincide with the integers, the set $A_{n,r}$ is composed of some residue classes modulo n , a set which we call $\mathbf{a}_{n,r}$. Thus we can understand the problem of estimating the density of R_{i+1} within $(r \bmod Q_i)$ as sieving the integers by multiple residue classes to moduli in \mathcal{N}_{i+1} , a set of moduli whose prime factors are constrained to lie in $[P_i, P_{i+1})$. This is the situation treated by the Lovász-type Theorem, Theorem 4 above, and so, if we are able to solve the relevant fixed-point problem then we obtain that the fiber is non-empty. Note that in the initial stage, all of the sieving sets have size 0 or 1, so that in this stage we can appeal to the optimal Shearer-type Theorem, Theorem 2.

In practice we will not estimate the density of R_{i+1} over all of R_i , but only within certain ‘good’ fibers above a subset $R_i^* \subset R_i \bmod Q_i$. We will be deliberately vague at this point about the requirements of a good fiber. Roughly these ensure that the corresponding fixed-point problem has a favorable solution. Also, we require that $R_i^* \subset R_{i-1}^* \cap R_i$ so that the good sets are nested. We let $R_0^* = R_0 = \mathbb{Z}$.

For $i \geq 1$ we weight the set $\mathbb{Z}/Q_i\mathbb{Z}$ with a probability measure μ_i supported on $R_{i-1}^* \cap R_i$, chosen so as to guarantee that a large proportion of the fibers are good. The measure μ_1 is uniform on the set $R_0^* \cap R_1 = R_1 \subset \mathbb{Z}/Q_1\mathbb{Z}$,

$$\forall r \in R_0^* \cap R_1 \bmod Q_1, \quad \mu_1(r) = \frac{1}{|R_1 \bmod Q_1|}.$$

Taking the measure μ_i as given, define, for $i \geq 1$,

$$\pi_{\text{good}}(i) = \frac{\mu_i(R_i^*)}{\mu_i(R_{i-1}^* \cap R_i)}$$

to be the proportion of good fibers. For $i \geq 1$ and $r \in R_i^* \cap R_{i+1} \bmod Q_{i+1}$ we set

$$\mu_{i+1}(r) = \frac{\mu_i(r \bmod Q_i)}{\pi_{\text{good}}(i) |R_{i+1} \cap (r \bmod Q_i) \bmod Q_{i+1}|}.$$

Thus, for a fixed $r \in R_i$, μ_{i+1} is constant on $R_{i+1} \cap (r \bmod Q_i)$. That μ_i is a sequence of probability measures follows from [5] Lemma 2, although, note that the factor of $\frac{1}{\pi_{\text{good}}(i)}$ is not included in the definition of μ_i in [5], so that the measures there do not have mass 1. Throughout, when we write $\mathbf{E}_{r \in R_{i-1}^* \cap R_i}$ we mean expectation with respect to the measure μ_i .

Along with the measure μ_i we track some bias statistics of $R_{i-1}^* \cap R_i$. Let $\ell_k(m)$ be the multiplicative function given at prime powers by

$$\ell_k(p^j) = (j+1)^k - j^k.$$

For $i \geq 1$, the k th bias statistic of $R_{i-1}^* \cap R_i$ is defined to be

$$\beta_k^k(i) = \sum_{m|Q_i} \ell_k(m) \max_{b \bmod m} \mu_i((b \bmod m)).$$

The importance of the bias statistics is that they control moments of (mixtures of) the sizes of the sets $\mathbf{a}_{n,r}$ as r varies in $R_{i-1}^* \cap R_i$.

Lemma 5. *Let $i \geq 1$. Let $\{w_n : n \in \mathcal{N}_{i+1}\}$ be any collection of non-negative weights, not all of which are zero. For each $k \geq 1$ we have*

$$\mathbf{E}_{r \in R_{i-1}^* \cap R_i} \left(\sum_{n \in \mathcal{N}_{i+1}} w_n |\mathbf{a}_{n,r} \bmod n| \right)^k \leq \left(\sum_{n \in \mathcal{N}_{i+1}} w_n \right)^k \beta_k^k(i).$$

Proof. See Lemmas 4 and 5 of [5]. □

In addition to the bias statistics, it will be useful for us to track maximum biases among the various good fibers. Let $i \geq 0$ and let $n \in \mathcal{N}_{i+1}$. We define the maximum bias at n to be

$$b_n = \max_{r \in R_i^*} \max_{b \bmod n} \frac{|n R_{i+1} \cap (r \bmod Q_i) \cap (b \bmod n) \bmod Q_{i+1}|}{|R_{i+1} \cap (r \bmod Q_i) \bmod Q_{i+1}|}.$$

Note that these appeared only implicitly in [5], but to get a better quantitative bound it will be useful for us to track them more carefully here.

The iterative growth of the bias statistics $\beta_k(i)$ to $\beta_k(i+1)$ is controlled by the proportion of good fibers $\pi_{\text{good}}(i)$ and the maximal biases at $n \in \mathcal{N}_{i+1}$.

Lemma 6. *Let $i \geq 1$. For each $k \geq 1$ we have the bound*

$$\beta_k^k(i+1) \leq \frac{\beta_k^k(i)}{\pi_{\text{good}}(i)} \left(1 + \sum_{n \in \mathcal{N}_{i+1}} \frac{\ell_k(n) b_n}{n} \right).$$

Proof. This follows by tracing the proof of Proposition 3 of [5]. □

We now turn to giving a detailed account of Theorem 4.

4. THE LOCAL LEMMA AND GOOD FIBERS

Our Theorem 4, which is used to estimate the density of good fibers, is derived from the following improved version of the Lovász Local Lemma due to [1], see also [8].

Theorem 7 (Clique Lovász Local Lemma). *Suppose that $G = (V, E)$ is a dependency graph for family of events $\{A_v\}_{v \in V}$, each with probability $\mathbf{P}(A_v) \leq \pi_v$. Let N_v be the neighborhood of $v \in V$. Suppose that there exists sequence $\underline{\mu} = \{\mu_v\}_{v \in V}$ of reals in $[0, \infty)$ such that, for each $v \in V$,*

$$(7) \quad \mu_v \geq \pi_v \phi_v(\underline{\mu})$$

where

$$\phi_v(\underline{\mu}) = \sum_{\substack{R \subset \{v\} \cup N_v \\ R \text{ indep. in } G}} \prod_{v' \in R} \mu_{v'}.$$

Then

$$(8) \quad \mathbf{P} \left(\bigcap_{v \in V} A_v^c \right) \geq \exp \left(- \sum_{v \in V} \mu_v \right)$$

and, for all $U \subset V$,

$$(9) \quad \frac{\mathbf{P} \left(\bigcap_{v \in V} A_v^c \right)}{\mathbf{P} \left(\bigcap_{u \in U} A_u^c \right)} \geq \exp \left(- \sum_{v \in V \setminus U} \mu_v \right).$$

Remark. In the definition of ϕ_v , $R = \emptyset$ is to be included, with associated product equal to 1.

Proof. This theorem with conclusion

$$(10) \quad \mathbf{P} \left(\bigcap_{v \in V} A_v^c \right) \geq \prod_{v \in V} (1 - \pi_v)^{\phi_v(\underline{\mu}) - \mu_v}$$

is proven in [1], and the corresponding relative conclusion

$$(11) \quad \frac{\mathbf{P} \left(\bigcap_{v \in V} A_v^c \right)}{\mathbf{P} \left(\bigcap_{u \in U} A_u^c \right)} \geq \prod_{v \in V \setminus U} (1 - \pi_v)^{\phi_v(\underline{\mu}) - \mu_v}$$

follows directly from the argument there. To deduce (8) and (9), observe that

$$\phi_v(\underline{\mu}) - \mu_v \leq (1 - \pi_v) \phi_v(\underline{\mu}),$$

so that

$$(1 - \pi_v)^{\phi_v - \mu_v} \geq \exp(\phi_v(1 - \pi_v) \log(1 - \pi_v)) \geq \exp(-\phi_v \pi_v) \geq \exp(-\mu_v).$$

□

Recall that Theorem 4 applies in the context of a congruence system to moduli in a set \mathcal{N} , whose prime factors lie in a set \mathcal{P} . Each modulus $n \in \mathcal{N}$ has a set of residues \mathbf{a}_n , considered to be a probabilistic event with probability $\frac{|\mathbf{a}_n|}{n}$. We require a system of non-negative weights $\{x_p\}_{p \in \mathcal{P}}$ satisfying

$$x_p \geq \sum_{n \in \mathcal{N}: p|n} \frac{|\mathbf{a}_n \bmod n| \prod_{p'|n} (1 + x_{p'})}{n}$$

and the conclusion is that the uncovered set R has density at least

$$\mathbf{P}(R) \geq \exp \left(- \sum_{n \in \mathcal{N}} \frac{|\mathbf{a}_n \bmod n| \prod_{p|n} (1 + x_p)}{n} \right)$$

and that, for any $n \in \mathcal{N}$, for any $b \bmod n$,

$$\frac{\mathbf{P}(R \cap (b \bmod n))}{\mathbf{P}(R)} \leq \frac{\exp \left(\sum_{p|n} x_p \right)}{n}.$$

Deduction of Theorem 4. To deduce Theorem 4 from Theorem 7 we take V to be the set of non-trivial square-free products of primes in \mathcal{P} ,

$$V = \{v > 1, \text{square-free}, p|v \Rightarrow p \in \mathcal{P}\}.$$

The event associated to $v \in V$ is the union of congruences $(\mathbf{a}_n \bmod n)$ for which $\text{sqf}(n) = v$, and this event has probability

$$\pi_v = \sum_{n: \text{sqf}(n)=v} \frac{|\mathbf{a}_n|}{n}.$$

The dependency graph connects v_1 and v_2 if and only if $\text{GCD}(v_1, v_2) > 1$.

We take the weight μ_v to be multiplicative, $\mu_v = \pi_v \prod_{p|v} (1 + x_p)$. This has the effect of reducing (7) at v to the constraint

$$(12) \quad \prod_{p|v} (1 + x_p) \geq \phi_v(\underline{\mu}).$$

Notice that

$$\phi_v(\underline{\mu}) = \sum_{\substack{R \subset \{v\} \cup N_v \\ \text{independent}}} \prod_{v' \in R} \mu_{v'} \leq \prod_{p|v} \left(1 + \sum_{v': p|v'} \mu_{v'} \right)$$

since each term in the sum on the left appears in the expansion of the product on the right. Thus if we make the condition that for each $p|Q'$,

$$x_p \geq \sum_{v': p|v'} \mu_{v'},$$

which is the condition (12) in the case $v = p$, then (12) holds automatically for all v . In this way we have reduced to guaranteeing the system of prime constraints

$$(13) \quad \forall p|Q', \quad x_p \geq \sum_{v': p|v'} \pi_{v'} \prod_{p': p'|v'} (1 + x_{p'}),$$

which is the constraint of Theorem 4.

The first conclusion, (8) of Theorem 7 now gives that

$$\begin{aligned} \mathbf{P} \left(\bigcap_{n \in \mathcal{N}} (\mathbf{a}_n \bmod n)^c \right) &\geq \exp \left(- \sum_{v \in V} \pi_v \prod_{p|v} (1 + x_p) \right) \\ &= \exp \left(- \sum_{n \in \mathcal{N}} \frac{|\mathbf{a}_n \bmod n| \prod_{p|n} (1 + x_p)}{n} \right), \end{aligned}$$

which is the first conclusion of Theorem 4. To get the second conclusion, use

$$\begin{aligned} \frac{\mathbf{P}(R \cap (b \bmod n))}{\mathbf{P}(R)} &\leq \frac{\mathbf{P}\left((b \bmod n) \cap \bigcap_{n' \in \mathcal{N}, (n, n')=1} (\mathbf{a}_{n'} \bmod n')^c\right)}{\mathbf{P}\left(\bigcap_{n' \in \mathcal{N}} (\mathbf{a}_{n'} \bmod n')^c\right)} \\ &= \frac{1}{n} \frac{\mathbf{P}\left(\bigcap_{n' \in \mathcal{N}, (n, n')=1} (\mathbf{a}_{n'} \bmod n')^c\right)}{\mathbf{P}\left(\bigcap_{n' \in \mathcal{N}} (\mathbf{a}_{n'} \bmod n')^c\right)} \\ &\leq \frac{1}{n} \exp \left(\sum_{n' \in \mathcal{N}: (n', n) > 1} \frac{|\mathbf{a}_{n'} \bmod n'| \prod_{p|n'} (1 + x_p)}{n'} \right). \end{aligned}$$

The last term is bounded by

$$\frac{1}{n} \exp \left(\sum_{p|n} \sum_{n': p|n'} \frac{|\mathbf{a}_{n'} \bmod n'| \prod_{p'|n'} (1 + x_{p'})}{n'} \right) \leq \frac{1}{n} \exp \left(\sum_{p|n} x_p \right).$$

□

We now give a sufficient criterion to guarantee a good solution to the fixed point equation governing existence of weights in Theorem 4. Recall that we define

$$G_p(\underline{x}) = \sum_{n \in \mathcal{N}: p|n} \frac{|\mathbf{a}_n \bmod n| \prod_{p'|n} (1 + x_{p'})}{n}.$$

A trivial lower bound for a fixed point $\underline{G}(\underline{x}^{\text{fix}}) = \underline{x}^{\text{fix}}$ is

$$\underline{x}^0, \quad x_p^0 = \frac{G_p(\underline{0})}{1 - G_p(\underline{0})},$$

and we wish to say that a fixed point lies near \underline{x}^0 . The n th derivative $D^n \underline{G}(\underline{0})$ is a multilinear map $\bigotimes^n \ell^2(\mathcal{P}) \rightarrow \ell^2(\mathcal{P})$. Give it the usual operator norm,

$$\|D^n \underline{G}(\underline{0})\|_{\text{op}} = \sup_{\|v_1\|_{\ell^2} = \dots = \|v_n\|_{\ell^2} = 1} \|D^n \underline{G}(\underline{0})(v_1, \dots, v_n)\|_{\ell^2}.$$

The following standard but somewhat technical theorem guarantees that there exists such a fixed point $\underline{x}^{\text{fix}}$ close to \underline{x}^0 when there is good control of the operator norms of the derivatives of $D^n(\underline{G})(\underline{0})$ of \underline{G} at $\underline{0}$.

Theorem 8. *With the notation as above, let $M > 0$ be a parameter. Assume that*

$$B_\infty = \|\underline{G}(\underline{0})\|_{\ell^\infty} < 1,$$

and set $B_{2,0} = \|\underline{x}^0\|_{\ell^2}$ and

$$B_{\text{op}}(M) = \|D\underline{G}(\underline{0}) - \text{diag}(D\underline{G}(\underline{0}))\|_{\text{op}} + \sum_{n=2}^{\infty} \frac{M^{n-1}}{(n-1)!} \|D^n \underline{G}(\underline{0})\|_{\text{op}} < \infty.$$

Suppose that $\theta = \frac{B_{\text{op}}}{1-B_\infty} < 1$ and that $\frac{B_{2,0}}{1-\theta} \leq M$. Then there exists $\underline{x}^{\text{fix}} = \underline{x}^0 + \underline{\epsilon}$, $\underline{\epsilon} \geq \underline{0}$ solving the fixed point equation $\underline{G}(\underline{x}^{\text{fix}}) = \underline{x}^{\text{fix}}$, such that

$$\|\underline{\epsilon}\|_{\ell^2} \leq \frac{B_{2,0}\theta}{1-\theta}.$$

Proof. Let $F(\underline{x}) = \underline{G}(\underline{x}) - \underline{x}$ so that we seek to solve $F(\underline{x}^{\text{fix}}) = \underline{0}$. This we can attempt via ‘Newton’s method’.

Let

$$\mathcal{D} = \text{diag}(D\underline{G}(\underline{0})) = \text{diag}(G_p(\underline{0})), \quad \mathcal{O}\mathcal{D} = D\underline{G}(\underline{0}) - \text{diag}(D\underline{G}(\underline{0})).$$

Starting from the initial guess \underline{x}^0 as above, set $\underline{x}^{i+1} = \underline{x}^i + (I - \mathcal{D})^{-1}F(\underline{x}^i)$. We may also set $\underline{x}^{-1} = \underline{0}$, which is consistent with this definition. A moment's thought shows that the sequence \underline{x}^i is increasing, so that if it is bounded it converges to the desired fixed point, and $\underline{\epsilon} = \sum_{i=0}^{\infty} (\underline{x}^{i+1} - \underline{x}^i)$. Plainly $\|(I - \mathcal{D})^{-1}\|_{\text{op}} = \frac{1}{1-B_{\infty}}$, so that

$$\|\underline{x}^{i+1} - \underline{x}^i\|_{\ell^2} \leq \frac{1}{1-B_{\infty}} \|F(\underline{x}^i)\|_{\ell^2}.$$

Note that F is a polynomial. Thus

$$F(\underline{x}^i) = \sum_{k=0}^{\infty} \frac{D^k F(\underline{0})(\underline{x}^i, \dots, \underline{x}^i)}{k!}.$$

In this sum, write $DF(\underline{0}) = (\mathcal{D} - I) + \mathcal{O}\mathcal{D}$, and recall that $\underline{x}^i = \underline{x}^{i-1} + (I - \mathcal{D})^{-1}F(\underline{x}^{i-1})$, so that

$$DF(\underline{0})\underline{x}^i = \mathcal{O}\mathcal{D}\underline{x}^i + (\mathcal{D} - I)\underline{x}^{i-1} - F(\underline{x}^{i-1}).$$

On Taylor expanding $F(\underline{x}^{i-1})$ we find

$$(14) \quad F(\underline{x}^i) = \mathcal{O}\mathcal{D}(\underline{x}^i - \underline{x}^{i-1}) + \sum_{k=2}^{\infty} \frac{1}{k!} (D^k F(\underline{0})(\underline{x}^i, \dots, \underline{x}^i) - D^k F(\underline{0})(\underline{x}^{i-1}, \dots, \underline{x}^{i-1})).$$

Now we impose the constraint $\|\underline{x}^j\|_{\ell^2} \leq M$, which holds for $j = 1$, and which we will verify for all j by induction. With this assumption, by the usual trick with the triangle inequality in which we change one coordinate at a time,

$$\|D^k F(\underline{0})(\underline{x}^i, \dots, \underline{x}^i) - D^k F(\underline{0})(\underline{x}^{i-1}, \dots, \underline{x}^{i-1})\|_{\ell^2} \leq kM^{k-1} \|D^k F(\underline{0})\|_{\text{op}} \|\underline{x}^i - \underline{x}^{i-1}\|_{\ell^2},$$

so that $\|F(\underline{x}^i)\|_{\ell^2} \leq B_{\text{op}} \|\underline{x}^i - \underline{x}^{i-1}\|$ and

$$\|\underline{x}^{i+1} - \underline{x}^i\|_{\ell^2} \leq \frac{B_{\text{op}}}{1-B_{\infty}} \|\underline{x}^i - \underline{x}^{i-1}\|_{\ell^2} = \theta \|\underline{x}^i - \underline{x}^{i-1}\|_{\ell^2}.$$

Since

$$\|\underline{x}^0 - \underline{x}^{-1}\|_{\ell^2} = \|\underline{x}^0\|_{\ell^2} = B_{2,0},$$

we have $\|\underline{x}^i\| \leq \frac{B_{2,0}}{1-\theta} \leq M$ for all i , which verifies the condition above. It follows that $\|\underline{\epsilon}\|_{\ell^2} \leq \frac{B_{2,0}\theta}{1-\theta}$. \square

4.1. Random Lovász weights. For each $i = 1, 2, \dots$ on (a subset of good) fibers above $R_{i-1}^* \cap R_i$ we apply Theorem 4 with moduli $\mathcal{N} = \mathcal{N}_{i+1}$ and residues $\mathbf{a}_n = \mathbf{a}_{n,r}$. Thus we think of the quantities from Theorems 4 and 8 as depending upon the random variable r , e.g. $\underline{G}(\underline{0}) = \underline{G}(\underline{0}, r)$, $B_{\infty} = B_{\infty,r}$, $\underline{x}^0 = \underline{x}^0(r)$. We wish to understand properties of the distribution of $\underline{x}^{\text{fix}}(r)$, but will instead define good fibers in terms of ℓ^p control of $\underline{G}(\underline{0}, r)$, and control of $B_{\text{op}}(M, r)$, the other quantities of interest being controlled in terms of these. We now work to control B_{op} .

We directly verify that the partial derivatives of $D^k G$ are given by

$$D_{p_1} \dots D_{p_k} G_p = \begin{cases} \sum_{\substack{n \in \mathcal{N} \\ p, p_1 \dots p_k | k}} \frac{|\mathbf{a}_n \bmod n|}{n} \prod_{\substack{p' | n \\ p' \notin \{p_1, \dots, p_k\}}} (1 + x_{p'}) & \text{if } p_1, \dots, p_k \text{ distinct} \\ 0 & \text{otherwise} \end{cases}.$$

A simple bound for the operator norm of $D^k \underline{G}(\underline{0})$ is

$$\|D^k \underline{G}(\underline{0})\|_{\text{op}} \leq \|D^k \underline{G}(\underline{0})\|_{\ell^2} = \left(\sum_{\substack{p_1, \dots, p_k \\ \text{distinct}}} \|D_{p_1} \dots D_{p_k} \underline{G}(\underline{0})\|_{\ell^2}^2 \right)^{\frac{1}{2}},$$

which, in view of the evaluation of $D^k G$, is given by in $k \geq 2$

$$\|D^k \underline{G}(\underline{0})\|_{\text{op}}^2 \leq S_k$$

$$S_k := \sum_{p_1, \dots, p_k \text{ distinct}} \left(k \left(\sum_{\substack{n \in \mathcal{N} \\ p_1 \dots p_k | n}} \frac{|\mathbf{a}_n \bmod n|}{n} \right)^2 + \sum_{p \notin \{p_1, \dots, p_k\}} \left(\sum_{\substack{n \in \mathcal{N} \\ pp_1 \dots p_k | n}} \frac{|\mathbf{a}_n \bmod n|}{n} \right)^2 \right).$$

In $\mathcal{O}\mathcal{D} = D\underline{G}(\underline{0}) - \text{diag}(D\underline{G}(\underline{0}))$ the diagonal terms $p = p_1$ are missing, so that we recover the bound

$$\|\mathcal{O}\mathcal{D}\|_{\text{op}}^2 \leq \sum_{p \neq p_1} \left(\sum_{\substack{n \in \mathcal{N} \\ pp_1 | n}} \frac{|\mathbf{a}_n \bmod n|}{n} \right)^2 =: S_1.$$

By Cauchy-Schwarz, for positive weights W_1, W_2, W_3, \dots

$$B_{\text{op}}(M)^2 \leq \left(\sum_{k=1}^{\infty} \frac{W_k M^{k-1}}{(k-1)!} \right) \left(\sum_{k=1}^{\infty} \frac{M^{k-1}}{(k-1)!} \frac{S_k}{W_k} \right).$$

Let $\min(\mathcal{P}) \geq P+1$. We choose

$$W_1^2 = \frac{1}{(P \log P)^2}, \quad \forall k \geq 2, \quad W_k^2 = \frac{k}{(P \log P)^k},$$

from which it follows

$$\begin{aligned} B_{\text{op}}(M)^2 &\leq \left(\frac{1}{P \log P} + \sum_{k=2}^{\infty} \frac{k^{\frac{1}{2}} M^{k-1}}{(k-1)! (P \log P)^{\frac{k}{2}}} \right) \\ &\quad \times \left((P \log P) S_1 + \sum_{k=2}^{\infty} \frac{M^{k-1} (P \log P)^{\frac{k}{2}}}{k^{\frac{1}{2}} (k-1)!} S_k \right) \\ (15) \quad &=: \mathcal{C} \times \mathcal{S}. \end{aligned}$$

We record bounds for $\|\underline{G}(\underline{0})\|_{\ell^2}$ and S_1, S_2, \dots etc averaged over $r \in R_{i-1}^* \cap R_i$.

Lemma 9. *Let $i \geq 1$. For $r \in R_{i-1}^* \cap R_i$ consider $\mathcal{N} = \mathcal{N}_{i+1}$ and $\mathbf{a}_n = \mathbf{a}_{n,r}$ as in the discussion above. We have the following bounds.*

$$\begin{aligned} \mathbf{E}_{r \in R_{i-1}^* \cap R_i} \|\underline{G}(\underline{0}, r)\|_{\ell^2}^2 &\leq \beta_2^2(i) \prod_{P_i \leq p < P_{i+1}} \left(1 + \frac{1}{p-1} \right)^2 \sum_{P_i \leq p < P_{i+1}} \frac{1}{(p-1)^2} \\ \mathbf{E}_{r \in R_{i-1}^* \cap R_i} \|\underline{G}(\underline{0}, r)\|_{\ell^3}^3 &\leq \beta_3^3(i) \prod_{P_i \leq p < P_{i+1}} \left(1 + \frac{1}{p-1} \right)^3 \sum_{P_i \leq p < P_{i+1}} \frac{1}{(p-1)^3} \\ \mathbf{E}_{r \in R_{i-1}^* \cap R_i} S_1(r) &\leq \beta_2^2(i) \prod_{P_i \leq p < P_{i+1}} \left(1 + \frac{1}{p-1} \right)^2 \left(\sum_{P_i \leq p < P_{i+1}} \frac{1}{(p-1)^2} \right)^2 \end{aligned}$$

and, for $k \geq 2$,

$$\begin{aligned} \mathbf{E}_{r \in R_{i-1}^* \cap R_i} S_k(r) &\leq \beta_2^2(i) \prod_{P_i \leq p < P_{i+1}} \left(1 + \frac{1}{p-1}\right)^2 \\ &\quad \times \left[k \left(\sum_{P_i \leq p < P_{i+1}} \frac{1}{(p-1)^2} \right)^k \left(1 + \frac{1}{k} \sum_{P_i \leq p < P_{i+1}} \frac{1}{(p-1)^2} \right) \right] \end{aligned}$$

Proof. These follow directly from the convexity lemma, Lemma 5, and the bound, for distinct $P_i \leq p_1, \dots, p_k < P_{i+1}$,

$$\sum_{\substack{n \in \mathcal{N}_{i+1} \\ p_1 \dots p_k | n}} \frac{1}{n} < \frac{1}{(p_1-1) \dots (p_k-1)} \sum_{n \in \mathcal{N}_{i+1}} \frac{1}{n} < \frac{1}{(p_1-1) \dots (p_k-1)} \prod_{P_i \leq p < P_{i+1}} \left(1 + \frac{1}{p-1}\right).$$

□

Inserting (15) in the last lemma, we conclude the following bound.

Lemma 10. *Let B_{op} be the constant from Theorem 8. Averaged over $R_{i-1}^* \cap R_i$, we have the bound*

$$\begin{aligned} \mathbf{E}_{r \in R_{i-1}^* \cap R_i} B_{\text{op}}(M)^2 &\leq \mathcal{C}_i \beta_2^2(i) \prod_{P_i \leq p < P_{i+1}} \left(1 + \frac{1}{p-1}\right)^2 \\ &\quad \times \left(P_i \log P_i \left(\sum_{P_i \leq p < P_{i+1}} \frac{1}{(p-1)^2} \right)^2 \right. \\ &\quad \left. + \left(1 + \frac{1}{P_i}\right) \sum_{n=2}^{\infty} \frac{n^{\frac{1}{2}} M^{n-1} (P_i \log P_i)^{\frac{n}{2}}}{(n-1)!} \left(\sum_{P_i \leq p < P_{i+1}} \frac{1}{(p-1)^2} \right)^n \right) \end{aligned}$$

with \mathcal{C}_i given as above by

$$\mathcal{C}_i = \left(\frac{1}{P_i \log P_i} + \sum_{n=2}^{\infty} \frac{n^{\frac{1}{2}} M^{n-1}}{(n-1)! (P_i \log P_i)^{\frac{n}{2}}} \right).$$

We conclude this section with a brief discussion of how we apply Theorem 4. Beyond demonstrating that fibers above a good set R_i^* are non-empty, the information that we wish to obtain from Theorem 4 is a bound for the bias statistics $\beta_k(i+1)$ in the next stage of iteration. Lemma 6 reduces this problem to bounding the individual biases b_n of $R_i^* \cap R_{i+1}$ at $n \in \mathcal{N}_{i+1}$, and Theorem 4 demonstrates that this bias is bounded by

$$(16) \quad b_n \leq \max_{r \in R_i^*} \exp \left(\sum_{p|n} x_p^{\text{fix}}(r) \right).$$

We bound this quantity in terms of the number of prime factors $\omega = \omega(n)$ of n . Thinking of $\underline{\epsilon} = \underline{\epsilon}(r)$ as a small error, we have

$$\sum_{p|n} x_p^{\text{fix}}(r) \leq \|\underline{x}^{\text{fix}}\|_{\infty, \omega} \leq \|\underline{x}^0\|_{\infty, \omega} + \sqrt{\omega} \|\underline{\epsilon}\|_{\ell^2},$$

where $\|\cdot\|_{\infty, k}$ denotes the norm

$$\|\underline{x}\|_{\infty, \omega} = \max_{i_1 < i_2 < \dots < i_\omega} (|x_{i_1}| + \dots + |x_{i_\omega}|).$$

Since we typically have information regarding $\|\underline{G}(\underline{0})\|_{\ell^p}$ for $p = 2$ or 3 (or both) we are led to a maximization problem of the type,

$$(17) \quad \begin{aligned} &\text{given:} && 0 < B_p < 1, \ 1 \leq \omega \\ &\text{maximize:} && \|\underline{x}^0\|_{\infty, \omega} \\ &\text{subject to:} && \|\underline{G}(\underline{0})\|_{\ell^p} \leq B_p. \end{aligned}$$

In the case $p = 2$ this may be easily solved along the following lines. It is no loss to assume that $\underline{G}(\underline{0}) \in \mathbb{R}_{\geq 0}^\omega$. An application of Lagrange multipliers gives that the coordinates of the optimum take at most 3 values, $0 = c_1 < c_2 \leq \frac{1}{3} \leq c_3 \leq B_2$, subject to $c_2(1 - c_2)^2 = c_3(1 - c_3)^2$. When there are two non-zero values, c_2 is constrained by $c_2(1 - c_2)^2 \geq B_2(1 - B_2)^2$, which is only possible for a bounded number of non-zero entries. For large ω , the optimum is $\frac{\sqrt{\omega}B_2}{1 - \frac{B_2}{\sqrt{\omega}}}$, so that the best choice for all ω is a finite check.

The case $p = 3$ is actually simpler, because, in that case also there are at most 2 non-zero values, and they necessarily satisfy $c_2 = 1 - c_3$.

5. EXPLICIT CALCULATION IN INITIAL STAGES

In the initial stage, we appeal to the Shearer-type Theorem, Theorem 3, with the primes in the range $P_0 = 4 < p < P_1 = 222$ and we verify numerically that the condition of the theorem holds. We also calculate the bound for bias statistics

$$\beta_2(1) \leq 12.25, \quad \beta_3(1) \leq 25.$$

The method of performing these explicit computations is described in Section A.

Let $P_2 = 4000$.

In order to choose the good set $R_1^* \subset R_1 (= R_0^* \cap R_1)$ we appeal to Lemmas 9 and 10 to calculate, for any $C_2, C_{\text{op}} > 0$, and for $M = 1.769746269$,

$$\begin{aligned} &\mathbf{E}_{r \in R_1} (C_2 \|\underline{G}(\underline{0}, r)\|_{\ell^2}^2 + C_{\text{op}} B_{\text{op}} (1.769746269, r)^2) \\ &\leq C_2 \beta_2^2(1) \prod_{222 \leq p < 4000} \left(1 + \frac{1}{p-1}\right)^2 \sum_{222 \leq p < 4000} \frac{1}{(p-1)^2} \\ &+ C_{\text{op}} \mathcal{C}_1 \beta_2^2(1) \prod_{222 \leq p < 4000} \left(1 + \frac{1}{p-1}\right)^2 \\ &\times \left[(222 \log 222) \sum_{222 \leq p < 4000} \frac{1}{(p-1)^2} \right. \\ &\left. + \frac{223}{222} \sum_{n=2}^{\infty} \frac{n^{\frac{1}{2}} 1.769746269^{n-1} (2 \cdot 222^2 \log 222)^{\frac{n}{2}}}{(n-1)!} \left(\sum_{222 \leq p < 4000} \frac{1}{(p-1)^2} \right)^n \right] \end{aligned}$$

and

$$\mathcal{C}_1 = \frac{1}{222(\log 222)} + \sum_{n=2}^{\infty} \frac{n^{\frac{1}{2}} 1.769746269^{n-1}}{(n-1)! (222 \log 222)^{\frac{n}{2}}}.$$

We calculate numerically that

$$\mathbf{E}_{r \in R_1} (C_2 \|\underline{G}(\underline{0}, r)\|_{\ell^2}^2 + C_{\text{op}} B_{\text{op}} (M, r)^2) < C_2 \cdot 0.246514091 + C_{\text{op}} \cdot 0.002220166.$$

We choose $C_2 = \frac{0.9}{0.246514091}$ and $C_{\text{op}} = \frac{0.1}{0.002220166}$, so that the above inequality reads

$$\mathbf{E}_{r \in R_1} (C_2 \|\underline{G}(\underline{0}, r)\|_{\ell^2}^2 + C_{\text{op}} B_{\text{op}} (1.769746269, r)^2) < 1.$$

We say that $r \in R_1$ is good if

$$\|C_2 \underline{G}(\underline{0}, r)\|_{\ell^2}^2 + C_{\text{op}} B_{\text{op}}(1.769746269, r)^2 \leq \frac{1}{1 - 0.3}.$$

By Markov's inequality, $\pi_{\text{good}}(1) \geq 0.3$.

Evidently, for all r ,

$$\|\underline{G}(\underline{0}, r)\|_{\ell^2}^2 \leq \frac{1}{C_2(1 - 0.3)} < 0.391292208.$$

We save a little extra ground by conditioning on the actual size of $\|\underline{G}(\underline{0}, r)\|_{\ell^2}$. Let $K = 100$ be a parameter. For $1 \leq j \leq K$ we say that $r \in R_1^*$ is in bin \mathcal{B}_j if

$$\|\underline{G}(\underline{0}, r)\|_{\ell^2}^2 \in \left(\frac{j-1}{K}, \frac{j}{K} \right] \cdot \frac{1}{C_2(1 - 0.3)}.$$

For $r \in \mathcal{B}_j$ we have

$$B_{\text{op}}(1.769746269, r)^2 \leq \frac{K - j + 1}{K} \cdot \frac{1}{C_{\text{op}}(1 - 0.3)}.$$

Abusing notation, for $r \in \mathcal{B}_j$ we write quantities depending upon r as depending upon j instead so, we use $\underline{G}(\underline{0}, j)$, $B_2(j)$, $B_{\text{op}}(j)$, and so forth.

In each bin we update

$$B_{\infty}(j) = \|\underline{G}(\underline{0}, j)\|_{\ell^{\infty}} \leq B_2(j)$$

and thus

$$B_{2,0}(j) = \|\underline{x}^0\|_{\ell^2} \leq \frac{B_2(j)}{1 - B_2(j)},$$

and $\theta(j) \leq \frac{B_{\text{op}}(j)}{1 - B_2(j)}$. We check numerically, bin-by-bin, that for all bins,

$$\frac{B_{2,0}(j)}{1 - \theta(j)} < 1.769746269 = M$$

so that the condition of Theorem 8 is met. In particular, each good fiber is non-empty.

Again, we apply Theorem 8 bin-by-bin so that, in each bin we obtain a bound of

$$\|\underline{\epsilon}(j)\|_{\ell^2} \leq \frac{B_{2,0}(j)\theta(j)}{1 - \theta(j)}.$$

Beginning from the information $B_2^2(j) \leq \frac{j}{K} 0.625533539$, we solve the optimization problem (17) for $\omega = 1, 2, 3, \dots$. As we have already commented, for each ω the optimal solution has no more than two non-zero values among the x_p . When it has the two values c_1, c_2 these satisfy for some positive integers $a \leq b$, $a + b \leq \omega$,

$$ac_1^2 + bc_2^2 \leq \frac{j}{K} 0.391292208, \quad c_1(1 - c_1)^2 = c_2(1 - c_2)^2.$$

It transpires that this possibility occurs only for $\omega \leq 4$ and when $a = 1$. For $\omega \geq 5$, the optimum in each bin is given by

$$\|\underline{x}^0(j)\|_{\infty, \omega} \leq \omega \frac{B_2(j)}{\sqrt{\omega} - B_2(j)}, \quad \|\underline{x}^{\text{fix}}(j)\| \leq \omega \frac{B_2(j)}{\sqrt{\omega} - B_2(j)} + \sqrt{\omega} \|\underline{\epsilon}(j)\|_{\ell^2}.$$

Obviously $B_2(j) \leq B_2(K) \leq 0.391292208^{\frac{1}{2}} = 0.625533539$, and we find

$$\sup_j \|\underline{\epsilon}(j)\|_{\ell^2} \leq 0.292129153.$$

Resulting bounds for $\sup_j \|\underline{x}^{\text{fix}}\|_{\infty, \omega}$ are recorded in the following table

ω	$\ \underline{x}^{\text{fix}}\ _{\infty, \omega}$
1	1.769746269
2	1.900670975
3	2.033321919
4	2.184489901
5	2.363269323
6	2.530235874
7	2.686345986
8	2.833661687
9	2.973253326
10	3.106051540
$\omega > 10$	$\omega \frac{0.625533539}{\sqrt{\omega}-0.625533539} + 0.292129153\sqrt{\omega}$

We can thus update the bound for bias statistics $\beta_k(2)$ according to Lemma 6. Write, for $k = 1, 2, 3, \dots$,

$$\tau_k(p) = \sum_{i=1}^{\infty} \frac{(i+1)^k - i^k}{p^i}.$$

for the local factor at p that occurs at the k th bias statistic. Then the new bound becomes

$$\beta_k^k(2) \leq \frac{\beta_k^k(1)}{\pi_{\text{good}}(1)} \left(1 + \sum_{j=1}^{\infty} \exp(\|\underline{x}^{\text{fix}}\|_{\infty, j}) e_j(\tau_k(p) : P_1 \leq p < P_2) \right)$$

where e_j indicates the j th elementary symmetric function. For large j we use the bound $e_j(\underline{\tau}) \leq \frac{e_1(\underline{\tau})^j}{j!}$. In this way we calculate that

$$\beta_2(2) < 94.66051416, \quad \beta_3(2) < 199.2834489.$$

6. ASYMPTOTIC ESTIMATES

Recall that $P_2 = 4000$. For all $i \geq 2$ we let $P_{i+1} = P_i^{1.5}$. In this section we use the following explicit estimates for sums and products over primes, which hold for $i \geq 2$.

$$\begin{aligned} \prod_{P_i \leq p < P_{i+1}} \left(\frac{p}{p-1} \right) &< (1.004212)(1.5) = 1.506318 \\ \sum_{P_i \leq p < P_{i+1}} \frac{1}{(p-1)^2} &< \frac{1.002631}{P_i \log P_i} \\ \sum_{P_i \leq p < P_{i+1}} \frac{1}{(p-1)^3} &< \frac{1.004382}{2P_i^2 \log P_i}. \end{aligned}$$

These are verified in Appendix B.

For the remainder of the argument our inductive assumption is, for $i \geq 2$,

$$\begin{aligned} (18) \quad \beta_2(i) &\leq 0.5197033883 \cdot (P_i \log P_i)^{\frac{1}{2}} \\ \beta_3(i) &\leq 0.3100980448 \cdot (2P_i^2 \log P_i)^{\frac{1}{3}}. \end{aligned}$$

Note that both of these hold at $i = 2$.

Setting $M = 2.949873427$ in Theorem 8, we estimate

$$\mathbf{E}_{r \in R_{i-1}^* \cap R_i} (C_3 \|\underline{G}(\underline{0}, r)\|_{\ell^3}^3 + C_2 \|\underline{G}(\underline{0}, r)\|_{\ell^2}^2 + C_{\text{op}} B_{\text{op}}(2.949873427, r)^2).$$

Appealing to Lemma 10 we bound the sums in \mathcal{C} and $\mathbf{E}\mathcal{S}$, implicitly defined in (15), by

$$\begin{aligned}\mathcal{C}_i &\leq \mathcal{C}_2 = \left(\frac{1}{4000 \log 4000} + \sum_{n=2}^{\infty} \frac{n^{\frac{1}{2}} M^{n-1}}{(n-1)!(4000 \log 4000)^{\frac{n}{2}}} \right) \leq 0.0001571422884, \\ \mathbf{E}\mathcal{S}_i &\leq (1.506318)^2 \beta_2^2(i) \left(\frac{(1.002631)^2}{P_i \log P_i} + \left(1 + \frac{1}{P_i}\right) \sum_{n=2}^{\infty} \frac{n^{\frac{1}{2}} M^{n-1} (1.002631)^n}{(n-1)!(P_i \log P_i)^{\frac{n}{2}}} \right) \\ &\leq (1.506318)^2 (0.5197033883)^2 \left((1.002631)^2 + \frac{4001}{4000} \sum_{n=2}^{\infty} \frac{n^{\frac{1}{2}} M^{n-1} (1.002631)^n}{(n-1)!(4000 \log 4000)^{\frac{n}{2}-1}} \right) \\ &\leq 3.212501212.\end{aligned}$$

Combined with the asymptotics of $\mathbf{E}\|\underline{G}(\underline{0})\|_{\ell^p}^p$ from Lemma 9,

$$\begin{aligned}\mathbf{E}_{r \in R_{i-1}^* \cap R_i} \|\underline{G}(\underline{0})\|_{\ell^3}^3 &\leq \prod_{P_i \leq p < P_{i+1}} \left(\frac{p}{p-1} \right)^3 \beta_3^3(i) \sum_{P_i \leq p < P_{i+1}} \frac{1}{(p-1)^3} \\ &< (1.506318)^3 (1.004382) (0.3100980448)^3 < 0.1023637064 \\ \mathbf{E}_{r \in R_{i-1}^* \cap R_i} \|\underline{G}(\underline{0})\|_{\ell^2}^2 &\leq \prod_{P_i \leq p < P_{i+1}} \left(\frac{p}{p-1} \right)^2 \beta_2^2(i) \sum_{P_i \leq p < P_{i+1}} \frac{1}{(p-1)^2} \\ &< (1.506318)^2 (1.002631) (0.5197033883)^2 < 0.6144485964\end{aligned}$$

we deduce

$$\begin{aligned}\mathbf{E}_{r \in R_{i-1}^* \cap R_i} (C_3 \|\underline{G}(\underline{0})\|_{\ell^3}^3 + C_2 \|\underline{G}(\underline{0})\|_{\ell^2}^2 + C_{\text{op}} B_{\text{op}} (2.949873427, r)^2) \\ \leq 0.1023637064 C_3 + 0.6144485964 C_2 + 0.0005048197920 C_{\text{op}}.\end{aligned}$$

Choose $C_3 = \frac{0.7}{0.1023637064}$, $C_2 = \frac{0.2}{0.6144485964}$, $C_{\text{op}} = \frac{0.1}{0.0005048197920}$ so that the expectation is bounded by 1. As before, declare $r \in R_{i-1}^* \cap R_i$ to be good if

$$C_3 \|\underline{G}(\underline{0}, r)\|_{\ell^3}^3 + C_2 \|\underline{G}(\underline{0}, r)\|_{\ell^2}^2 + C_{\text{op}} B_{\text{op}} (2.949873427, r)^2 \leq \frac{1}{1-0.3}.$$

Evidently $\pi_{\text{good}}(i) \geq 0.3$, and for good r ,

$$\begin{aligned}\|\underline{G}(\underline{0}, r)\|_{\ell^\infty}^3 &\leq \|\underline{G}(\underline{0}, r)\|_{\ell^3}^3 \leq \frac{1}{C_3(1-0.3)} < 0.2089055233, \\ \|\underline{G}(\underline{0}, r)\|_{\ell^2}^2 &\leq \frac{1}{C_2(1-0.3)} < 4.388918546,\end{aligned}$$

but, again, we bin to get a stronger result.

For $K = 100$ and integers $0 < i, j$, $i+j \leq K+1$ let the bin $\mathcal{B}_{i,j}$ be those r for which

$$\|\underline{G}(\underline{0}, r)\|_{\ell^3}^3 \in \left(\frac{i-1}{K}, \frac{i}{K} \right] \frac{1}{C_3(1-0.3)}, \quad \|\underline{G}(\underline{0}, r)\|_{\ell^2}^2 \in \left(\frac{j-1}{K}, \frac{j}{K} \right] \frac{1}{C_2(1-0.3)}.$$

For $r \in \mathcal{B}_{i,j}$ we have

$$B_{\text{op}}^2(r) \leq \frac{K-i-j+1}{K} \frac{1}{C_{\text{op}}(1-0.3)}.$$

We proceed much as before (again replacing r with i, j in each argument) updating bin-by-bin

$$B_{2,0}(i, j) \leq \frac{\|\underline{G}(\underline{0}, i, j)\|_{\ell^2}}{1 - \|\underline{G}(\underline{0}, i, j)\|_{\ell^3}},$$

and

$$\theta(i, j) \leq \frac{B_{\text{op}}(2.949873427, i, j)}{1 - \|\underline{G}(\underline{Q}, i, j)\|_{\ell^3}}.$$

We check bin-by-bin that

$$\frac{B_{2,0}(i, j)}{1 - \theta(i, j)} < 2.949873427 = M$$

so that our choice of $M = 2.949873427$ in Theorem 8 is valid.

In each bin we solve the optimization problem (17) with $p = 3$, and we find that for all $\omega \geq 1$ and for all bins the optimum is

$$\|\underline{x}^0(i, j)\|_{\infty, \omega} \leq \omega^{\frac{2}{3}} \frac{\frac{i}{K} 0.5933577790}{1 - \frac{\frac{i}{K} 0.5933577790}{\omega^{\frac{1}{3}}}},$$

so that we guarantee

$$\|\underline{x}^{\text{fix}}(i, j)\|_{\infty, \omega} \leq \frac{\frac{i}{K} 0.5933577790 \omega^{\frac{2}{3}}}{1 - \frac{\frac{i}{K} 0.5933577790}{\omega^{\frac{1}{3}}}} + \|\underline{\epsilon}(i, j)\|_{\ell^2} \omega^{\frac{1}{2}}.$$

We calculate

$$\max_{i, j} \|\underline{\epsilon}(i, j)\|_{\ell^2} \leq 0.190000303.$$

Thus we find the following bounds.

ω	$\ \underline{x}^{\text{fix}}\ _{\infty, \omega}$
1	1.459164221
2	1.780349459
3	2.096937862
4	2.387653719
5	2.656941273
6	2.909180305
7	3.147611526
8	3.374605257
9	3.591932780
10	3.800951606
$\omega > 10$	$\frac{0.5933577790 \omega}{\omega^{\frac{1}{3}} - 0.5933577790} + 0.190000303 \sqrt{\omega}$

In Appendix B we verify that, for $i \geq 2$,

$$e_1(\tau_2(p)) \leq 3 \log 1.5 + 0.00334 < 1.21974, \quad e_1(\tau_3(p)) \leq 7 \log 1.5 + 0.00779 < 2.84605.$$

Hence,

$$e_j(\tau_2(p) : P_i \leq p < P_{i+1}) \leq \frac{e_1(\tau_2(p))^j}{j!} < \frac{(1.21974)^j}{j!}, \quad e_j(\tau_3(p)) < \frac{(2.84605)^j}{j!}$$

and we find

$$\frac{\beta_2^3(i+1)}{\beta_2^3(i)} \leq \frac{1}{0.3} \left(1 + \sum_{\omega=1}^{\infty} \exp \left(\max_{i, j} \|\underline{x}^{\text{fix}}(i, j)\|_{\infty, \omega} \right) \frac{(1.21974)^\omega}{\omega!} \right) < 48.515$$

and

$$\frac{\beta_3^3(i+1)}{\beta_3^3(i)} \leq \frac{1}{0.3} \left(1 + \sum_{\omega=1}^{\infty} \exp \left(\max_{i, j} \|\underline{x}^{\text{fix}}(i, j)\|_{\infty, \omega} \right) \frac{(2.84605)^\omega}{\omega!} \right) < 487.17.$$

On the other hand,

$$\frac{P_{i+1} \log P_{i+1}}{P_i \log P_i} \geq 1.5 \cdot P_i^{\frac{1}{2}} \geq 1.5 \cdot 4000^{\frac{1}{2}} > 94$$

and

$$\frac{P_{i+1}^2 \log P_{i+1}}{P_i^2 \log P_i} \geq 1.5 \cdot P_i \geq 6000,$$

so that (18) is preserved, which completes the proof by induction.

APPENDIX A. SYMMETRIC FUNCTIONS

We briefly describe how we performed the calculations in the initial stage of the argument, see Section 5. There we appealed to Theorem 3, which is Theorem 2 with set $[n]$ identified with $\{p : 4 < p < 222\} = p_1 < p_2 < \dots < p_n$ and weights $\pi_p = \frac{1}{p-1}$. We identify square-free number m with the set of its prime factors. The Shearer functions

$$\rho(p_1) > \rho(p_1 p_2) > \dots > \rho(p_1 \dots p_n)$$

are easily computed via

$$\rho(p_1 \dots p_j) = \sum_{i=0}^j X(i) e_i(\pi_{p_1}, \dots, \pi_{p_j}).$$

with the e_i elementary symmetric functions (take $e_0 = 1$), see [10].

The bias statistics are also not difficult to bound. Recall that $Q = \text{LCM}(m : m \in \mathcal{M})$ and that Q_1 is the part of Q composed of primes less than $P_1 = 222$. The k th bias statistic is

$$\beta_k^k(1) = \sum_{m|Q_1} \ell_k(m) \max_{b \bmod m} \frac{|R_1 \cap (b \bmod m) \bmod Q_1|}{|R_1 \bmod Q_1|}.$$

Let $\text{sqf}(m) = \prod_{p \in S} p =: m_S$. Appealing to (4) of Theorem 3 we have

$$\frac{|R_1 \cap (b \bmod m) \bmod Q_1|}{|R_1 \bmod Q_1|} \leq \frac{1}{m} \frac{\rho([n] \setminus S)}{\rho([n])},$$

so that

$$\begin{aligned} \beta_k^k(1) &\leq \frac{1}{\rho([n])} \sum_{S \subset [n]} \rho([n] \setminus S) \sum_{m: \text{sqf}(m)=m_S} \frac{\ell_k(m)}{m} \\ (19) \quad &\leq \frac{1}{\rho([n])} \sum_{S \subset [n]} \rho([n] \setminus S) \prod_{s \in S} \left(\sum_{j=1}^{\infty} \frac{\ell_k(p_s^j)}{p_s^j} \right). \end{aligned}$$

Recall that we define

$$\tau_{k,i} = \tau_k(p_i) = \sum_{j=1}^{\infty} \frac{\ell_k(p_i^j)}{p_i^j} = \sum_{j=1}^{\infty} \frac{(j+1)^k - j^k}{p_i^j} = \left[\left(\frac{1}{x} - 1 \right) \left(x \frac{\partial}{\partial x} \right)^k \frac{1}{1-x} - 1 \right]_{x=\frac{1}{p_i}}.$$

Define for $i+j \leq n$ the mixed symmetric functions $f_{i,j}(\underline{\pi}, \underline{\tau}_k)$ by

$$f_{i,j}(\underline{\pi}, \underline{\tau}_k) = \frac{i!j!(n-i-j)!}{n!} \sum_{\sigma \in \text{Sym}([n])} \pi_{\sigma(1)} \dots \pi_{\sigma(i)} \tau_{k,\sigma(i+1)} \dots \tau_{k,\sigma(i+j)},$$

or, equivalently, by

$$\sum_{0 \leq i+j \leq n} f_{i,j}(\underline{\pi}, \underline{\tau}_k) x^i y^j = \prod_{i=1}^n (1 + x\pi_i + y\tau_{k,i}).$$

The sum of (19) is a linear combination of the mixed symmetric functions $f_{i,j}(\underline{\pi}, \underline{\tau}_k)$

$$\sum_{S \subset [n]} \rho([n] \setminus S) \prod_{s \in S} \tau_{k,s} = \sum_{0 \leq i+j \leq n} X(i) f_{i,j}(\underline{\pi}, \underline{\tau}_k),$$

and so is rapidly computable.

APPENDIX B. EXPLICIT PRIME NUMBER ESTIMATES

In this appendix we sketch proofs for explicit bounds on well-known prime sums and products. Recall $P_2 = 4000$ and, for $i \geq 2$, $P_{i+1} = P_i^{1.5}$. In particular, no P_i is prime. Let γ denote the Euler-Mascheroni constant. Dusart [2], Theorem 6.12 proves the following estimate.

Theorem 11. *For $x > 1$ we have*

$$e^\gamma(\log x) \left(1 - \frac{0.2}{(\log x)^2}\right) < \prod_{p \leq x} \frac{p}{p-1}$$

and, for $x \geq 2973$ we have

$$\prod_{p \leq x} \frac{p}{p-1} < e^\gamma(\log x) \left(1 + \frac{0.2}{(\log x)^2}\right).$$

As a consequence, we obtain

Corollary 12. *For $i \geq 2$ we have*

$$\prod_{P_i \leq p < P_{i+1}} \left(\frac{p}{p-1}\right) < (1.5)(1.004212).$$

For the sums of reciprocals of squares of primes, we have the following estimate.

Proposition 13. *For $i \geq 2$,*

$$\sum_{P_i \leq p < P_{i+1}} \frac{1}{(p-1)^2} < \frac{1.002631}{P_i \log P_i},$$

and

$$\sum_{P_i \leq p < P_{i+1}} \frac{1}{(p-1)^3} < \frac{1.004382}{2P_i^2 \log P_i}.$$

Proof. We prove only the first inequality, as the second is similar. One easily checks

$$\sum_{P_2 \leq p < P_3} \frac{1}{(p-1)^2} < \frac{1}{P_2 \log P_2}, \quad \sum_{P_3 \leq p < P_4} \frac{1}{(p-1)^2} < \frac{1}{P_3 \log P_3}.$$

For $i \geq 4$ use $\frac{0.99999997}{(p-1)^2} < \frac{1}{p^2}$ so that, for $x \geq P_4$,

$$0.99999997 \sum_{p \geq x} \frac{1}{(p-1)^2} < \sum_{p \geq x} \frac{1}{p^2} = -\frac{\theta(x)}{x^2 \log x} + \int_x^\infty \frac{\theta(y)}{y^3} \frac{1+2 \log y}{(\log y)^2} dy.$$

By [2], we have the inequality $|\theta(x) - x| < 0.2 \frac{x}{(\log x)^2}$ for $x \geq 3594641$. In particular, $0.99913x < \theta(x) < 1.00088x$ in this range. Also, Lemma 9 of [7] yields

$$\int_x^\infty \frac{1+2 \log y}{y^2 (\log y)^2} dy < \frac{2}{x \log x}.$$

Combined, these estimates give the claim. □

Recall that we define $\tau_k(x) = \sum_{i=1}^\infty \frac{(i+1)^k - i^k}{x^i}$. We have

$$\tau_2(x) = \frac{3x-1}{(x-1)^2}, \quad \tau_3(x) = \frac{7x^2-2x+1}{(x-1)^3}.$$

Proposition 14. For $i \geq 2$,

$$\sum_{P_i \leq p < P_{i+1}} \tau_2(p) < 3 \log 1.5 + 0.00334,$$

and

$$\sum_{P_i \leq p < P_{i+1}} \tau_3(p) < 7 \log 1.5 + 0.00779.$$

Proof. For $i = 2, 3$ this is verified directly. For $x \geq P_4$ this is a consequence of the following estimate of [2], Theorem 6.11. \square

Theorem 15. There is a constant B , such that, for $x > 10372$,

$$\left| \sum_{p \leq x} \frac{1}{p} - \log \log x - B \right| \leq \frac{1}{10(\log x)^2} + \frac{4}{15(\log x)^3}.$$

APPENDIX C. THEOREMS OF LOVÁSZ AND SHEARER-TYPE

The Lovász local lemma considers the following scenario. In a probability space \mathcal{X} there are events $\{A_v\}_{v \in V}$ with dependency graph $G = (V, E)$, that is, A_v is independent of the σ -algebra $\sigma(A_w : (v, w) \notin E)$. One seeks a positive lower bound for $\mathbf{P}(\bigcap_{v \in V} \overline{A_v})$. The local lemma guarantees that if there exist weights $1 > x_v \geq \mathbf{P}(A_v)$ satisfying

$$\forall v \in V, \quad x_v \prod_{w: (v, w) \in E} (1 - x_w) \geq \mathbf{P}(A_v)$$

then

$$\mathbf{P}\left(\bigcap_{v \in V} \overline{A_v}\right) \geq \prod_{v \in V} (1 - x_v).$$

Shearer [9] gives an optimal bound of the above type via the independent set polynomial

$$\Xi(z_v : v \in V) = 1 + \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{\substack{(v_1, \dots, v_n) \in V^n \\ \forall i \neq j, v_i \sim v_j}} z_{v_1} \dots z_{v_n}.$$

where $(v_i \sim v_j)$ means $v_i \neq v_j$ and $(v_i, v_j) \notin E$.

Theorem (Shearer's Theorem). Given $S \subset V$ let $\Xi_S(z_v : v \in V)$ denote Ξ with arguments $z_v : v \in S$ replaced by 0. Subject to

$$\forall S \subset V, \quad \Xi_S(-\mathbf{P}(A_v) : v \in V) \geq 0$$

it holds

$$\mathbf{P}\left(\bigcap_{v \in V} \overline{A_v}\right) \geq \Xi(-\mathbf{P}(A_v) : v \in V).$$

Although Shearer's Theorem is tight, evaluating the independent set polynomial is difficult and so there remains interest in finding statements of a similar type to the local lemma, which is more easily applied.

One way to reduce the complexity of Shearer's theorem is to organize the events into collection of cliques. We consider the scenario in which graph $G = (V, E)$ is covered by a collection of cliques \mathcal{K} , that is, $E = \bigcup_{K \in \mathcal{K}} E_K$. For $v \in V$, let

$$(20) \quad p(v) = \{K : v \in K\}.$$

We make the assumption that $p(v)$ uniquely determines v and we assume that all vertices have self-loops. What is the same, we take $V = \mathcal{P}(\mathcal{K}) \setminus \{\emptyset\}$ to be the collection of all

non-empty subsets of \mathcal{K} , and, for $S_1, S_2 \in V$, set $(S_1, S_2) \in E$ if and only if $S_1 \cap S_2 \neq \emptyset$. Consider vertex variables $(z_v)_{v \in V}$ and clique variables $(\theta_K)_{K \in \mathcal{K}}$. For $v \in V$, set also $\theta_v = \prod_{K: v \in K} \theta_K$. The clique partition function is defined to be

$$\Xi(\mathbf{v}, \boldsymbol{\theta}) = 1 + \sum_{n \geq 1} \frac{1}{n!} \sum_{\substack{v_1, \dots, v_n \\ \text{indep. in } G}} z_{v_1} \dots z_{v_n} \theta_{v_1} \dots \theta_{v_n}.$$

Evidently $\Xi(\mathbf{v}, \boldsymbol{\theta})$ specializes to $\Xi(\mathbf{v})$ at $\boldsymbol{\theta} = 1$. Using this, we prove a clique version of Shearer's theorem.

Theorem 16 (Clique Shearer Theorem). *Let events $\{A_v : v \in V\}$ in probability space \mathcal{X} have dependency graph $G = (V, E)$ covered by cliques \mathcal{K} as above. For $S \subset \mathcal{K}$ define event $B_S = \bigcup_{v: p(v) \subset S} A_v$. Subject to the condition*

$$\forall S \subset \mathcal{K}, \quad \Xi(-\mathbf{P}(A_v), 1_S) > 0$$

we have for all $\emptyset \subset S \subset T \subset \mathcal{K}$,

$$\mathbf{P}(\overline{B_T} | \overline{B_S}) \geq \frac{\Xi(-\mathbf{P}(A_v), 1_T)}{\Xi(-\mathbf{P}(A_v), 1_S)}.$$

Remark. As compared to Shearer's Theorem, the clique Shearer Theorem has the advantage that the number of conditions which must be checked is exponential in the number of cliques, rather than in the number of vertices.

Proof. The proof is by induction. Let $S \subset \mathcal{K}$ and suppose the conclusion holds for subsets $T \subset S$. Let $K \in \mathcal{K} \setminus S$. Then

$$\begin{aligned} \frac{\mathbf{P}(\overline{B_{S \cup \{K\}}})}{\mathbf{P}(\overline{B_S})} &\geq 1 - \sum_{\substack{w: K \in p(w) \\ p(w) \subset S \cup \{K\}}} \frac{\mathbf{P}(A_w \cap \overline{B_S})}{\mathbf{P}(\overline{B_S})} \\ &\geq 1 - \sum_{\substack{w: K \in p(w) \\ p(w) \subset S \cup \{K\}}} \frac{\mathbf{P}(A_w) \mathbf{P}(\overline{B_{S \setminus p(w)}})}{\mathbf{P}(\overline{B_S})} \\ &\geq 1 - \sum_{\substack{w: K \in p(w) \\ p(w) \subset S \cup \{K\}}} \mathbf{P}(A_w) \frac{\Xi(-\mathbf{P}(A_v), 1_{S \setminus p(w)})}{\Xi(-\mathbf{P}(A_v), 1_S)} \\ &= \frac{\Xi(-\mathbf{P}(A_v), 1_{S \cup \{K\}})}{\Xi(-\mathbf{P}(A_v), 1_S)}. \end{aligned}$$

□

As a consequence we obtain a proof of a generalization of Theorem 3.

Theorem (Shearer-type theorem). *Suppose we have a probability space and a parameter $\theta \geq 1$. Let $[n] = \{1, 2, \dots, n\}$, and assume that for each $1 \leq i \leq n$ there is a weight π_i assigned, satisfying $\frac{1}{\theta} \geq \pi_1 \geq \pi_2 \geq \dots \geq \pi_n \geq 0$. Let the sets $\emptyset \neq T \subset [n]$ index events A_T each having probability*

$$0 \leq \mathbf{P}(A_T) \leq \theta \prod_{t \in T} \pi_t := \pi_T.$$

Assume that A_T is independent of $\sigma(\{A_S : S \subset [n], S \cap T = \emptyset\})$, so that a valid dependency graph for the events $\{A_T : \emptyset \neq T \subset [n]\}$ has an edge between $S \neq T$ whenever $S \cap T \neq \emptyset$.

Define $\rho_\theta(\emptyset) = 1$, and given $\emptyset \neq T \subset [n]$, set

$$\rho_\theta(T) = 1 - \sum_{\emptyset \neq S_1 \subset T} \pi_{S_1} + \sum_{\substack{\emptyset \neq S_1, S_2 \subset T \\ S_1 < S_2 \text{ disjoint}}} \pi_{S_1} \pi_{S_2} - \sum_{\substack{\emptyset \neq S_1, S_2, S_3 \subset T \\ S_1 < S_2 < S_3 \text{ disjoint}}} \pi_{S_1} \pi_{S_2} \pi_{S_3} + \dots$$

Suppose that $\rho_\theta([1]) \geq \rho_\theta([2]) \geq \dots \geq \rho_\theta([n]) > 0$. Then for any $\emptyset \neq T \subset [n]$,

$$\mathbf{P} \left(\bigcap_{\emptyset \neq S \subset T} A_S^c \right) \geq \rho_\theta(T) > 0$$

and, for any $T_1 \subset T_2 \subset [n]$,

$$\frac{\mathbf{P} \left(\bigcap_{\emptyset \neq S \subset T_2} A_S^c \right)}{\mathbf{P} \left(\bigcap_{\emptyset \neq S \subset T_1} A_S^c \right)} \geq \frac{\rho_\theta(T_2)}{\rho_\theta(T_1)}.$$

Proof. It is observed in [10] that $\rho_\theta(T)$ may be expressed as a linear combination of elementary symmetric functions in $\{\pi_t : t \in T\}$. Indeed, if $B(m, j)$ denotes the generalized Bell number, that is, the number of ways of partitioning a set of size m into j parts then, setting $|T| = M$ and making the convention $e_0(\underline{\pi}) = 1$,

$$\rho_\theta(T) = 1 + \sum_{m=1}^M \left(\sum_{j=1}^m (-\theta)^j B(m, j) \right) e_m(\underline{\pi}) := \sum_{i=0}^M X_\theta(i) e_i(\underline{\pi}),$$

where X_θ satisfies the recurrence

$$X_\theta(0) = 1, \quad \forall i \geq 1, \quad X_\theta(i) = -\theta \sum_{j=0}^{i-1} \binom{i-1}{j} X_\theta(j).$$

In particular, as exploited in [8], $\rho(T)$ is affine linear in each variable π_t .

We check that under the given conditions, $\rho_\theta(T) > 0$ for any $T \subset [n]$, which reduces this theorem to the clique Shearer Theorem.

Given vectors $\underline{x}, \underline{y} \in \mathbb{R}^m$, say that $\underline{x} \leq \underline{y}$ if $x_i \leq y_i$ for each i . By induction, we show that for any $1 \leq m \leq n$ and for $\underline{0} \leq \underline{x} \leq \underline{\pi}$, $\rho_\theta(\underline{x}) \geq \rho_\theta(\underline{\pi}) > 0$, from which the case for T follows since the π_i are decreasing.

When $m = 1$, $\rho_\theta(\pi_1) = 1 - \theta\pi_1 \leq 1 - \theta x_1 = \rho_\theta(x_1)$. Given $m > 1$, assume inductively the statement for all $m' < m$.

Note that, by hypothesis, we have $\rho_\theta(\pi_1, \dots, \pi_{m-1}) \geq \rho_\theta(\pi_1, \dots, \pi_m) > 0$.

We show by an inner induction that for $1 \leq j \leq m$,

$$\rho_\theta(x_1, \dots, x_j, \pi_{j+1}, \dots, \pi_m) \geq \rho_\theta(\pi_1, \dots, \pi_m).$$

When $j = 1$ this holds, since $(\pi_2, \dots, \pi_m) \leq (\pi_1, \dots, \pi_{m-1})$ so that, by the inductive assumption

$$\rho_\theta(\pi_2, \dots, \pi_m) \geq \rho_\theta(\pi_1, \dots, \pi_{m-1}) \geq \rho_\theta(\pi_1, \dots, \pi_m),$$

from which

$$\rho_\theta(x_1, \pi_2, \dots, \pi_m) \geq \rho_\theta(\pi_1, \dots, \pi_m)$$

follows by affine linearity.

Having shown

$$\rho_\theta(x_1, \dots, x_{j-1}, \pi_j, \dots, \pi_m) \geq \rho_\theta(\pi_1, \dots, \pi_m)$$

the case

$$\rho_\theta(x_1, \dots, x_j, \pi_{j+1}, \dots, \pi_m) \geq \rho_\theta(\pi_1, \dots, \pi_m)$$

again follows by affine linearity from

$$\begin{aligned}\rho_\theta(x_1, \dots, x_{j-1}, 0, \pi_{j+1}, \dots, \pi_m) &\geq \rho_\theta(\pi_1, \dots, \pi_{j-1}, 0, \pi_{j+1}, \dots, \pi_m) \\ &\geq \rho_\theta(\pi_1, \dots, \pi_{m-1}, 0) \\ &\geq \rho_\theta(\pi_1, \dots, \pi_m).\end{aligned}$$

□

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